B.Sc. Semester-II Examination, 2023

MATHEMATICS [Honours]

Course ID : 22112 **Course Code : SH/MTH/202/C-4**

Course Title : Differential Equations and Vector Calculus

[OLD SYLLABUS]

Time : 2 Hours Full Marks : 40

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meaning.

UNIT-I

1. Answer any **five** from the following questions: $2\times5=10$

a) Does the function $f(x,y)=\sin x+y^2$ satisfy a Lipschitz condition w.r.t. to y on the rectangle $R=\{(x,y): 0\le x\le 1, 0\le y\le 3\}$? Justify your answer.

b) Find the particular integral (P.I.) of the ODE: $F(D)y=f(x)$,

where $F(D)=(D-m_1)(D-m_2)....(D-m_n)$ and

$\left(D\equiv\frac{d}{dx}\right)$.

c) Evaluate : $\int_1^3\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt$, where $\vec{r}=t^3\hat{i}+2t\hat{j}-t^2\hat{k}$.

d) Find out the critical points of the system of differential equations:

$$\frac{dx}{dt}=y^2-5x+6,\ \frac{dy}{dt}=y-x.$$

e) If $\vec{\alpha}=t^2\hat{i}-t\hat{j}+(2t+1)\hat{k}$ and $\vec{\beta}=(2t-3)\hat{i}+\hat{j}-t\hat{k}$

then find $\frac{d}{dt}\left(\alpha\times\frac{d\beta}{dt}\right)$ at t=2.

f) Whether the following functions are linearly dependent or independent?

$$x^2-x+1,\ x^2-1,\ 3x^2-x-1.$$

g) Find the volume of the parallelopiped whose three concurrent edges are represented by the vectors

$\vec{a}=3\hat{\imath}-5\hat{\jmath}-4\hat{k},\quad \vec{b}=\hat{\imath}+3\hat{\jmath}-2\hat{k},$.

$\vec{c}=3\hat{\imath}+\hat{\jmath}-2\hat{k}$.

h) If $\frac{d\vec{a}}{dt}=\vec{\omega}\times\vec{a}$ and $\frac{d\vec{b}}{dt}=\vec{\omega}\times\vec{b}$, show that

$$\frac{d}{dt}(\vec{a}\times\vec{b})=\vec{\omega}\times(\vec{a}\times\vec{b}).$$

UNIT-II

2. Answer any **four** from the following questions:

$5\times4=20$

a) Solve that differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{e^x}{1+e^x}$$

by method of variation of parameters.

b) Find the general solution of the differential equation

$$\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = x^2 \cos x.$$

c) If $\frac{d^2x}{dt^2} = -n^2x$ at $t=0$, $x=a$, and $\frac{dx}{dt}=u$, find the particular solution. Hence prove that the maximum value of x is $\sqrt{a^2+\frac{u^2}{n^2}}$. 3+2

d) i) If $\vec{r} = \begin{cases} 2\hat{i}-\hat{j}+2\hat{k}, & when\ t=2 \\ 4\hat{i}-2\hat{j}+3\hat{k}, & when\ t=3 \end{cases}$, then find the value of $\int_2^3 \left(\vec{r}.\frac{d\vec{r}}{dt}\right)dt.$

ii) If $\vec{\alpha}\neq\vec{0}$ but $\vec{\alpha}.\vec{\beta}=\vec{\alpha}.\vec{\gamma}$ and $\vec{\alpha}\times\vec{\beta}=\vec{\alpha}\times\vec{\gamma}$ then prove that $\vec{\beta}=\vec{\gamma}$. 3+2

e) i) Prove that the necessary and sufficient condition for a vector $\vec{r}=\vec{f}(t)$ to have a constant direction is $\vec{f}\times\frac{d\vec{f}}{dt}=\vec{0}$

ii) Prove that

$$(\vec{\alpha}\times\vec{\beta}).(\vec{\gamma}\times\vec{\delta})+(\vec{\alpha}\times\vec{\gamma}).(\vec{\delta}\times\vec{\beta})+(\vec{\alpha}\times\vec{\delta}).(\vec{\beta}\times\vec{\gamma})=0.$$

3+2

f) Determine the nature of the equilibrium point of the linear system:

$$\frac{dx}{dt}=4x-y,$$

$$\frac{dy}{dt}=2x+y.$$

Also sketch the corresponding phase portrait in the phase plane. 2+3

UNIT-III

3. Answer any **one** of the following questions:

$10\times1=10$

a) i) Show that $x=0$ is an ordinary point of the differential equation

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x+2)y = 0.$$

Hence find the power series solution of the above differential equation about $x=0$.

ii) Prove that three non-zero non-collinear vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar if and only if $[\vec{a}\ \vec{b}\ \vec{c}] = 0$. (1+6)+3

b) i) Solve the following linear system by using nature of the characteristic roots:

$$\frac{dx}{dt} = x - 4y,$$

$$\frac{dy}{dt} = x + y.$$

ii) Find the particular solution of the ODE: $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 3\sin 2t$ which satisfies the conditions $x = 0, \frac{dx}{dt} = 0$ at $t = 0$.

iii) Prove that

$$|\vec{\alpha}\times\vec{\beta}|^2 |\vec{\alpha}\times\vec{\gamma}|^2 - \{(\vec{\alpha}\times\vec{\beta}).(\vec{\alpha}\times\vec{\gamma})\}^2 = |\vec{\alpha}|^2 [\vec{\alpha}\vec{\beta}\vec{\gamma}]^2.$$

4+3+3

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